HNN extensions of finite semilattices (Languages, Algebra and Computer Systems)

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Abstract

A finitely presented inverse semigroup is the most interesting object of research in inverse semigroup theory from the point of view of algorithmic problems. Several finitely presented inverse semigroups can be presented as HNN extensions of finite semilattices. In this paper we discuss such inverse semigroups.

1 Introduction

HNN extensions of semigroups were introduced by Howie [3] in a restricted case. A more general definition was given in [8] so that the class of HNN extensions can include important classes of inverse semigroups. For example, free inverse semigroups, free inverse monoids, free Clifford semigroups and the bicyclic semigroup have HNN extension structures. HNN extensions and amalgamated free products of inverse semigroups were employed to show the undecidability of Markov properties of inverse semigroups and the undecidability of some other decision problems in [8]. It seems that HNN extensions and amalgamated free products are indispensable tools to study algorithmic problems in inverse semigroup theory. In the present paper we investigate HNN extensions of finite semilattices as a first step toward understanding algebraic structures of HNN extensions of inverse semigroups. For more details on HNN extensions of semilattices, we refer the reader to [9].

Let S be an inverse semigroup, A_i and B_i $(i \in I)$ inverse subsemigroups of S. Suppose that $e_i \in A_i \subset e_iSe_i$, $f_i \in B_i \subset f_iSf_i$ for some idempotents e_i, f_i of S and that ϕ_i is an isomorphism of A_i onto B_i for every $i \in I$. Then the inverse semigroup S^* presented by

$$Inv(S, t_i\ (i \in I) \mid t_i^{-1}at_i = \phi_i(a)\ \forall a \in A_i,\ t_i^{-1}t_i = f_i,\ t_it_i^{-1} = e_i\ i \in I)$$

is called an *HNN extension of S associated with $\phi_i : A_i \to B_i$ $(i \in I)$*. Each element t_i in S^* is called a *stable letter.* A class $\mathbf{C}$ of semigroups is said to have the *weak HNN property* if $\mathbf{C}$ satisfies the following condition:

Suppose that S, A, $B \in \mathbf{C}$, $e \in A \subset eSe$, $f \in B \subset fSf$ for some $e, f \in E(S)$. Let $\phi : A \to B$ be an isomorphism. Then there exists $T \in \mathbf{C}$ and an embedding $\psi : S \hookrightarrow T$ such that $t'\psi(a)t = \psi(\phi(a))$ for all $a \in A$, $t't = \psi(f)$ and $tt' = \psi(e)$ for some $t \in T$ and $t' \in V(t)$.

Moreover, the class $\mathbf{C}$ is said to have the *strong HNN property* if $\mathbf{C}$ satisfies the following:

Suppose that $S, A, B \in \mathbf{C}$ and $A \subset eSe, B \subset fSf$ for some $e, f \in E(S)$. Let $\phi : A \to B$ be an isomorphism. There is $T \in \mathcal{C}$ and an embedding $\psi : S \hookrightarrow T$ such that $t'\psi(a)t = \psi(\phi(a)), tt' = \psi(e), t't = \psi(f)$ for some $t \in T$ and $t' \in V(t)$ and $t'\psi(S)t \cap \psi(S) = t'\psi(A)t = \psi(B)$.

It is shown that the class of inverse semigroups has the strong HNN property in [8]. Hence, an inverse semigroup S is naturally embedded into an HNN extension of S. We usually identify S and the isomorphic inverse subsemigroup of the HNN extension, and hence, we have

$$t_i^{-1}St_i \cap S = t_i^{-1}A_it_i = B_i$$

for each $i \in I$ in S^*.

2 HNN extensions of finite semilattices

There are many inverse semigroups which are presented as HNN extensions of finite semilattices. For example, free inverse semigroups on a finite set and the bicyclic semigroup are HNN extensions of finite semilattices as we see below.

Example Free inverse semigroups: Let $\{x\}$ be a singleton set and $FIS(\{x\})$ be the free inverse semigroup on $\{x\}$. Let E be the semilattice presented by

$$Inv(\{e, f, g\} \mid e^2 = e, f^2 = f, g^2 = g, ef = fe = g).$$

Clearly E is the free semilattice on two generators (e and f) and so E is a finite semilattice. Put $A = \{e\}$, $B = \{f\}$ and let $\phi : A \to B$ be the trivial isomorphism. Let $S = Inv(E, t \mid t^{-1}et = f, t^{-1}t = f, tt^{-1} = e)$. Clearly S is an HNN extension of E associated with ϕ. We show S is the free inverse semigroup generated by an singleton set using Tietze transformations. We have

$$Inv(E, t \mid t^{-1}et = f, t^{-1}t = f, tt^{-1} = e)$$

$$= Inv(e, f, g, t \mid e^2 = e, f^2 = f, g^2 = g,$$

$$ef = fe = g, t^{-1}et = f, t^{-1}t = f, tt^{-1} = e)$$

$$= Inv(e, f, g, t \mid (tt^{-1})^2 = tt^{-1}, (t^{-1}t)^2 = t^{-1}t, (t^{-1}ttt^{-1})^2 = t^{-1}ttt^{-1},$$

$$t^{-1}ttt^{-1} = tt^{-1}t^{-1}t = g, t^{-1}tt^{-1}t = t^{-1}t, t^{-1}t = f, tt^{-1} = e)$$

$$= Inv(t \mid (tt^{-1})^2 = tt^{-1}, (t^{-1}t)^2 = t^{-1}t, (t^{-1}ttt^{-1})^2 = t^{-1}ttt^{-1}, t^{-1}tt^{-1}t = t^{-1}t)$$

$$= Inv(t \mid \emptyset) = FIS(\{t\}).$$

A similar argument shows that the free inverse semigroup of rank n is an HNN extension of the free semilattice on $2n$ generators.

Example The Bicyclic semigroup: First of all we should note that the bicyclic semigroup B is presented by

$$Inv(x \mid xx^{-1}x^{-1}x = x^{-1}x).$$

Let $E = \{e_1, e_2\}$ be a two element semilattice such that $e_1 > e_2$. Put $A = \{e_1\}$ and $B = \{e_2\}$. Let $\phi : A \to B$ be the trivial isomorphism. Then the inverse semigroup S presented by $Inv(E, t \mid t^{-1}e_1t = e_2, t^{-1}t = e_2, tt^{-1} = e_1)$ is an HNN extension. Using Tietze transformations, we have

$$S = Inv(e_1, e_2, t \mid e_1^2 = e_1, e_2^2 = e_2,$$

$$e_1e_2 = e_2e_1 = e_2, t^{-1}e_1t = e_2, t^{-1}t = e_2, tt^{-1} = e_1)$$

$$= Inv(e_1, e_2, t \mid (tt^{-1})^2 = tt^{-1}, (t^{-1}t)^2 = t^{-1}t,$$

$$tt^{-1}t^{-1}t = t^{-1}ttt^{-1} = t^{-1}t, t^{-1}tt^{-1}t = t^{-1}t, t^{-1}t = e_2, tt^{-1} = e_1)$$

$$= Inv(t \mid (tt^{-1})^2 = tt^{-1}, (t^{-1}t)^2 = t^{-1}t,$$

$$tt^{-1}t^{-1}t = t^{-1}ttt^{-1} = t^{-1}t, t^{-1}tt^{-1}t = t^{-1}t)$$

$$= Inv(t \mid tt^{-1}t^{-1} = t^{-1}, ttt^{-1} = t) = B.$$

Hence, the bicyclic semigroup is the HNN extension of a finite semilattice.

Another important example is a universally E-unitary inverse semigroup. An inverse semigroup S is *universally E-unitary* if S is presented by

$$Inv(X \mid e_i = f_i \ (i = 1, 2, \cdots n))$$

where $X = \{x_1, x_2, \ldots, x_n\}$ and e_i and f_i are Dyck words on X. We refer the reader to [9] for the results and terminology on universally E-unitary inverse semigroups. The word problem for a universally E-unitary inverse monoid is considered by Margolis and Meakin [5]. Using Rabin's tree theorem, they showed the solvability of the word problem for a universally E-unitary inverse monoid. An alternate approach is provided in [7]. The similar result for inverse semigroups follows immediately.

Proposition 1 ([5]) *Let S be an inverse semigroup presented by*

$$Inv(X \mid e_i = f_i \ (i = 1, 2, \cdots n))$$

where $X = \{x_1, x_2, \ldots, x_n\}$ and e_i and f_i are Dyck words on X. Then the word problem for S is solvable.

We borrow some terminology and notation from language theory. Let L_1 and L_2 be subsets of X^* and Y^*, respectively. We define the *Shuffle product* of L_1 and L_2 to be the set

$$\{u_1v_1u_2v_2\dots u_nv_n \in (X \cup Y)^* \mid u_1u_2\dots u_n \in L_1,\ v_1v_2\dots v_n \in L_2,\ n \geq 1\}$$

and denote it by $L_1 \diamond L_2$.

Lemma 2 *Let $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_m\}$ and D the Dyck language on X. Suppose that e_i and f_i are in $(D \cup \{1\}) \diamond (Y \cup Y^{-1})^+$ for each $i = 1, 2, \dots, s$ where 1 denotes the empty word. Let S be the inverse semigroup presented by*

$$Inv(X,\ Y \mid e_i = f_i\ (i = 1, 2, \cdots s),\ y_ky_j = y_{\rho(k,j)})$$

where ρ is a function of $\{1, 2, \dots, m\} \times \{1, 2, \dots, m\}$ into $\{1, 2, \dots, m\}$ satisfying $\rho(k, k) = k$ and $\rho(k, j) = \rho(j, k)$. Then S has solvable word problem.

Proof. We consider the inverse semigroup S^* presented by

$$Inv(X,\ Y,\ t_y\ (y \in Y) \mid e_i = f_i\ (i = 1, 2, \cdots s),$$

$$y_ky_j = y_{\rho(k,j)},\ t_y^{-1}t_y = t_yt_y^{-1} = y\ \forall y \in Y).$$

We note that the inverse subsemigroup generated by Y in S is a semilattice since $y_iy_i = y_i$ for every $i = 1, 2, \cdots, n$. Clearly S^* is an HNN extension of S associated with the partial isomorphisms of E_y to E_y where $E_y = \{y\}$ for each $y \in Y$. Let e_i' and f_i' be words obtained from e_i and f_i by substituting $t_yt_y^{-1}$ for every $y \in Y$, respectively. We can easily check that e_i' and f_i' are

Dyck words on $X \cup \{t_y \mid y \in Y\}$. Using Tietze transformations of type II, we can show that S^* can be presented by

$$Inv(X,\ t_y\ (y \in Y) \mid R)$$

where R consists of

$$e'_i = f'_i\ (i = 1, 2, \cdots s), \quad t_{y_k} t_{y_k}^{-1} t_{y_j} t_{y_j}^{-1} = t_{y_{\rho(k,j)}} t_{y_{\rho(k,j)}}^{-1}, \quad t_y^{-1} t_y = t_y t_y^{-1}\ \forall y \in Y.$$

Then S^* has the presentation as in Proposition 1, and hence, S^* has solvable word problem. Since S^* is an HNN extension of S, S is embedded in S^*. Since S is finitely generated, S has solvable word problem. □

Theorem 3 *An HNN extension of a finite semilattice of finite non-idempotent rank has solvable word problem.*

Proof. Let S be an HNN extension of a semilattice E presented by

$$Inv(E,\ t_i\ (i \in I) \mid t_i^{-1} e t_i = \phi_i(e)\ \forall e \in E_i,\ t_i^{-1} t_i = f_i,\ t_i t_i^{-1} = e_i\ \forall i \in I)$$

where I is a finite set and $\phi_i : E_i \to F_i$ is an isomorphism for each $i \in I$. Assume that $E = \{h_j \mid j \in J\}$. Note that J is a finite set. Then we define the function σ of $J \times J$ into J by $\sigma(j_1, j_2) = j_3$ if and only if $h_{j_1} h_{j_2} = h_{j_3}$ in E. We note that σ satisfies the conditions $\sigma(k,k) = k$ and $\sigma(k,j) = \sigma(j,k)$. Clearly E is presented by

$$Inv(\{y_j \mid j \in J\} \mid R_1)$$

where R_1 consists of $y_{j_1}y_{j_2} = y_{\sigma(j_1,j_2)}$ with $j_1, j_2 \in J$. It follows that S is presented by

$$Inv(\{y_j \mid j \in J\},\ t_i\ (i \in I) \mid\ R_0,\ R_1)$$

where R_0 consists of $t_i^{-1}y(e)t_i = y(\phi_i(e))$ for all $e \in E_i$, $t_i^{-1}t_i = y(f_i)$ and $t_it_i^{-1} = y(e_i)$ for all $i \in I$ where $y(e)$ is an element in $\{y_j \mid j \in J\}$ corresponding to e in E. Therefore S has a presentation as in Lemma 2, and hence, the word problem for S is solvable. $\square$

For example, an HNN extension of a free inverse semigroup of finite rank associated with finite subsemilattices is an HNN extension of a finite semilattice of finite non-idempotent rank, and hence, it has solvable word problem. In general, it is shown that an HNN extension of a free inverse semigroup associated with finitely generated inverse subsemigroups has solvable word problem in [4]. It is also shown in [1] that a free product of free inverse semigroups amalgamating a finitely generated inverse subsemigroup has solvable word problem.

Examples: Let S_1, S_2 be the inverse semigroups presented by

$$Inv(x_1,\ x_2,\ y_1,\ y_2 \mid x_1^{-1}y_1y_2x_1 = y_2x_2x_2^{-1},$$

$$x_2^{-1}y_1x_2y_2 = y_1x_1x_1^{-1}y_2,\ y_1^2 = y_1,\ y_2^2 = y_2)$$

and

$$Inv(x_1,\ x_2,\ y_1,\ y_2,\ y_3 \mid x_1^{-1}y_1^{-1}y_2x_1 = x_2y_2^{-1}x_2^{-1}y_3,$$

$$y_1^2 = y_1,\ y_2^2 = y_2,\ y_1y_2 = y_3).$$

We should note that the presentations above are not the one as in Lemma 2, however, it is clear that S_1 and S_2 can be presented as in Lemma 2. Therefore both S_1 and S_2 have solvable word problem.

An HNN extension of a finite semilattice is finitely presented. Conversely, we can prove that finitely presented universally E-unitary inverse semigroup is an HNN extension of a finite semilattice. The proof of the next theorem is too long to put here and so we refer the reader to [9].

Theorem 4 **([9])** *If an inverse semigroup S is presented by*

$$Inv(X \mid e_i = f_i \ (i = 1, 2, \cdots m))$$

where $X = \{x_1, x_2, \cdots, x_n\}$ and e_i and f_i are Dyck words on X, then S is an HNN extension of a finite semilattice of non-idempotent rank n. □

We now raise a question whether or not every finitely generated universally E-unitary inverse semigroup is an HNN extension of a finite semilattice of finite non-idempotent rank. Let us see the following examples. Let $\mathbf{R}$ be a recursively enumerable and non-recursive set of non-negative integers. Then let S_1 and S_2 be the inverse semigroups presented by

$$Inv(x,\ y \mid x^{-r}x^r = y^{-r}y^r \ \forall r \in \mathbf{R})$$

and

$$Inv(x,\ t \mid t^{-1}x^{-r}x^r t = x^{-r}x^r \ \forall r \in \mathbf{R},\ t^{-1}t = tt^{-1} = x^{-m}x^m),$$

respectively, where m is the minimum number in $\mathbf{R}$. First of all, we note that S_1 and S_2 are universally E-unitary inverse semigroups because the defining relations are given by equations of Dyck words.

Lemma 5 *The inverse semigroups S_1 and S_2 defined above have unsolvable word problem.*

Proof. We show that $r \in \mathbf{R}$ if and only if $x^{-r}x^r = y^{-r}y^r$ in S_1 for non-negative integer r. We temporarily asssume that this is true. If the word problem for S_1 is solvable, then we can decide whether or not a non-negative integer r is in $\mathbf{R}$ using the algorithm that solves the word problem for S_1. This contradicts the fact that $\mathbf{R}$ is non-recursive. Hence, the word problem for S_1 is not solvable. We now show that $x^{-r}x^r = y^{-r}y^r$ in S_1 implies $r \in \mathbf{R}$. We note that S_1 is a free product of the free inverse semigroups $FIS(\{x\})$ and $FIS(\{y\})$ amalgamating the semilattices E_1 and E_2 where $E_1 = \{x^{-r}x^r \mid r \in \mathbf{R}\}$ and $E_2 = \{y^{-r}y^r \mid r \in \mathbf{R}\}$. We remark that E_1 and E_2 are chains of $FIS(\{x\})$ and $FIS(\{y\})$, respectively. We may regard $FIS(\{x\})$ and $FIS(\{y\})$ as subsemigroups of S_1. Obviously, $x^{-r}x^r \in E_1$ if and only if $r \in \mathbf{R}$ and $y^{-r}y^r \in E_2$ if and only if $r \in \mathbf{R}$. Since the class of inverse semigroups has the strong amalgamation property ([2]), $FIS(\{x\}) \cap FIS(\{y\}) = E_1 = E_2$ in S. Suppose that $x^{-r}x^r = y^{-r}y^r$ in S_1. Then $x^{-r}x^r = y^{-r}y^r \in FIS(\{x\}) \cap FIS(\{y\}) = E_1 = E_2$. Hence, we have $x^{-r}x^r \in E_1$ and so $r \in \mathbf{R}$. Conversely $r \in \mathbf{R}$ implies $x^{-r}x^r = y^{-r}y^r$ in S_1.

We note that S_2 is an HNN extension of the free inverse semigroup $FIS(\{x\})$ associated with the subsemilattice $\{x^{-r}x^r \mid r \in \mathbf{R}\}$. We can show that $t^{-1}x^{-r}x^r t = x^{-r}x^r$ if and only if $r \in \mathbf{R}$ in S_2 using the strong HNN property. Hence, S_2 also has unsolvable word problem. □

We note that the maximal group homomorphic images of S_1 and S_2 are the free group of rank 2 and so they have solvable word problem. It is interesting to ask whether or not there is a finitely presented E-unitary (or F-inverse) inverse semigroup whose word problem is unsolvable but its maximal group homomorphic image is a free group (or has solvable word problem).

Theorem 6 *There is a finitely generated universally E-unitary inverse semigroup which cannot be embedded into an HNN extension of a finite semilattice. In particular it is not finitely presented as an HNN extension of a finite semilattice, that is, it does not have a presentation as in Theorem 4.*

Proof. By Lemma 5, S_1 (or S_2) defined above has unsolvable word problem. If it is embedded in an HNN extension of a finite semilattice, then by Theorem 3 it has solvable word problem as S_1 (or S_2) is finitely generated. It follows that S_1 (or S_2) cannot be embedded in an HNN extension of a finite semilattice. □

The inverse semigroup S_1 (or S_2) defined above is recursively presented, nevertheless it cannot be embedded in a finitely presented universally E-unitary inverse semigroup. Therefore an analogue of Higman's embedding theorem in group theory does not hold for the class of universally E-unitary inverse semigroups.

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